is the system total mass. For the KL example the transfer terms are, from Eq. (1),

$$M\bar{\rho}_0 \times \dot{\bar{\rho}}_0 + m\bar{\rho}_1 \times \dot{\bar{\rho}}_1 \tag{5}$$

Using the KL notation, let the motion of mass m (point P) relative to the MC of main body A (point A^*) be defined by $\bar{q}$‡ where

$$\bar{q} = \bar{a}_1 q_1 + \bar{a}_2 q_2 \tag{6}$$

and where, referring to Fig. 1KL,

$$q_1 = b\sin\theta \qquad q_2 = c - r - b\cos\theta \tag{7}$$

From Eqs. (4)

$$\bar{\rho}_0 = -(\mu/M)\bar{q} \qquad \bar{\rho}_1 = (\mu/m)\bar{q} \tag{8}$$

where $\mu = mM/(M+m)$. Combining Eqs. (5) and (8) the total transfer term is $\mu\bar{q} \times \dot{\bar{q}}$ so that the total angular momentum with respect to the CMC is

$$\bar{H}_{\text{CMC}} = \bar{a}_1 I_1 \omega_1 + \bar{a}_2 I_2 \omega_2 + \bar{a}_3 I_3 \omega_3 + \mu\bar{q} \times \dot{\bar{q}} \tag{9}$$

Forming the inertial time derivative of $\bar{H}_{\text{CMC}}$ and equating this to the vector torque around the CMC produces Eqs. (10-12KL) directly. Equation (9KL) and its counterparts never appear.

2) A generalized angular momentum equation was derived by the author in a 1960 article[2] and subsequently applied in a 1962 paper.[3] An alternate derivation starting from first principles was later given in an internal set of notes.[4] Let point 0 be some reference point, moving or fixed, of any system of moving masses. Then with respect to this point the generalized equation is

$$\bar{L}_0 = \dot{\bar{H}}_0 + \bar{S}_0 \times \bar{a}_0 \tag{10}$$

In Eq. (10) $\bar{L}_0$ is the torque resulting from external forces, $\bar{H}_0$ is the system angular momentum, $\bar{S}_0$ is the system static moment (all evaluated with respect to point 0), and $\bar{a}_0$ is the inertial acceleration of point 0. For the KL example, take the reference point as the MC of the main body, point A^*. Evaluating all terms with respect to A^* and using the KL notation, actual or implied,

$$\bar{L}_{A^*} = \bar{T} + \bar{q} \times \bar{R} \qquad \bar{S}_{A^*} = m\bar{q}$$

$$\bar{H}_{A^*} = \bar{a}_1 I_1 \omega_1 + \bar{a}_2 I_2 \omega_2 + \bar{a}_3 I_3 \omega_3 + m\bar{q} \times \dot{\bar{q}} \tag{11}$$

The inertial acceleration of point A^* can be expressed in two ways: 1) kinematically, by introducing the velocity $\bar{v}^{A^*}$, and 2) dynamically, by introducing the external forces.

1) Since the inertial velocity is expressed in components along rotating axes, Eq. (3KL), then

$$\bar{a}^{A^*} = \bar{a}_1 \dot{v}_1 + \bar{a}_2 \dot{v}_2 + \bar{a}_3 \dot{v}_3 + \bar{\omega} \times \bar{v}^{A^*} \tag{12}$$

Substituting Eq. (12) for $\bar{a}^{A^*}$, combining with Eq. (11) for $\bar{S}_{A^*}$, and combining the result with $\bar{H}_{A^*}$ and $\bar{L}_{A^*}$ produces Eqs. (23-25KL).

2) Dynamically

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\bar{v}^{A^*} + m\bar{v}^P) = M\bar{a}^{A^*} + m\bar{a}^P = \bar{R} + \bar{S} \tag{13}$$

where $\bar{a}^P = \bar{a}^{A^*} + \ddot{\bar{q}}$, and where $\ddot{\bar{q}}$ is the inertial second derivative of $\bar{q}$ defined by Eqs. (6) and (7). Solving Eq. (13) for $\bar{a}^{A^*}$ and then combining with $\bar{S}_{A^*}$ and the result with $\bar{H}_{A^*}$ and $\bar{L}_{A^*}$ produces again Eqs. (10-12KL).

References

[1] Kane, T.R. and Levinson, D.A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, March-April 1980, pp. 99-112.

[2] Grubin, C., "On Generalization of the Angular Momentum Equation," *Journal of Engineering Education*, Vol. 51, 1960, p. 237.

[3] Grubin, C., "Dynamics of a Vehicle Containing Moving Parts," *Journal of Applied Mechanics*, Sept. 1962, pp. 486-488.

[4] Grubin, C., *Mechanics*, West Coast University, Los Angeles, Calif., 1967, Chap. 6.

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Reply by Authors to C. Grubin

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WE believe that the first of Dr. Grubin's points is factually incorrect and that the second is misleading. Accordingly, we shall attempt to set matters straight by discussing each point in turn.

1) Dr. Grubin's assertion that "by doing a few preliminary vector calculations, Eq. (10KL) can be derived directly rather than arriving at Eq. (9KL) and then ..." is invalid. This may be seen as follows.

The two equations at issue are

$$\begin{aligned} &I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 + \mu\langle (b\dot{\theta}\sin\theta - \dot{r})[\omega_1(c - r - b\cos\theta) \\ &\quad - \omega_2 b\sin\theta] + (c - r - b\cos\theta)[\dot{\omega}_1(c - r - b\cos\theta) \\ &\quad + \omega_1(b\dot{\theta}\sin\theta - \dot{r}) - \dot{\omega}_2 b\sin\theta - \omega_2\dot{\theta}b\cos\theta] \\ &\quad + \omega_2[b^2\dot{\theta} + b\sin\theta(\omega_3 b\sin\theta - \dot{r}) - (c - r)b\dot{\theta}\cos\theta \\ &\quad + (c - r - b\cos\theta)^2\omega_3] + \omega_3 b\sin\theta[\omega_1(c - r - b\cos\theta) \\ &\quad - \omega_2 b\sin\theta]\rangle = T_1 + \mu(c - r - b\cos\theta)(R_3/m - S_3/M) \end{aligned} \tag{9KL}$$

and

$$\begin{aligned} &I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 + \mu(c - r - b\cos\theta)[(\dot{\omega}_1 \\ &\quad + \omega_2\omega_3)(c - r - b\cos\theta) - (\dot{\omega}_2 - \omega_3\omega_1)b\sin\theta + 2b\dot{\theta}(\omega_1\sin\theta \\ &\quad - \omega_2\cos\theta) - 2\dot{r}\omega_1] = T_1 + \mu(c - r - b\cos\theta)(R_3/m - S_3/M) \end{aligned} \tag{10KL}$$

In our paper, it is shown that use of the Angular Momentum Principle leads directly to the first of these, and it is asserted that a considerable amount of labor is required to deduce the second from the first. Let us carry out the procedure which,

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according to Dr. Grubin, permits one to arrive at Eq. (10KL) "directly" after "doing a few preliminary vector calculations."

Following Dr. Grubin, one begins by forming the vector $\boldsymbol{q}$ in accordance with Eqs. (6G)‡ and (7G), which yields

$$\boldsymbol{q} = b\sin\theta \boldsymbol{a}_1 + (c-r-b\cos\theta)\boldsymbol{a}_2$$

Next, one differentiates $\boldsymbol{q}$ with respect to t in N (our Newtonian reference frame), obtaining (after some simplification)

$$\dot{\boldsymbol{q}} = [b\dot{\theta}\cos\theta - \omega_3(c-r-b\cos\theta)]\boldsymbol{a}_1 + (-\dot{r} + b\dot{\theta}\sin\theta + \omega_3 b\sin\theta)\boldsymbol{a}_2 + [\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]\boldsymbol{a}_3$$

so that one can now form $\boldsymbol{q}\times\dot{\boldsymbol{q}}$ as

$$\boldsymbol{q}\times\dot{\boldsymbol{q}} = (c-r-b\cos\theta)[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]\boldsymbol{a}_1 - b\sin\theta[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]\boldsymbol{a}_2 + [b^2\dot{\theta} + b\sin\theta(\omega_3 b\sin\theta - \dot{r}) - (c-r)b\dot{\theta}\sin\theta + \omega_3(c-r-b\cos\theta)^2]\boldsymbol{a}_3$$

Finally, one must substitute this expression into Eq. (9G), differentiate the resulting expression with respect to t in N, and equate the result to "the vector torque around the CMC." Now, differentiating $\boldsymbol{q}\times\dot{\boldsymbol{q}}$ with respect to t in N gives

$$\langle(-\dot{r} + b\dot{\theta}\sin\theta)[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta] + (c-r-b\cos\theta)[\dot{\omega}_1(c-r-b\cos\theta) + \omega_1(b\dot{\theta}\sin\theta - \dot{r}) - \dot{\omega}_2 b\sin\theta - \omega_2\dot{\theta}b\cos\theta] + \omega_2[b^2\dot{\theta} + b\sin\theta(\omega_3 b\sin\theta - \dot{r}) - (c-r)b\dot{\theta}\cos\theta + (c-r-b\cos\theta)^2\omega_3] + \omega_3 b\sin\theta[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]\rangle\boldsymbol{a}_1 + \langle\ldots\rangle\boldsymbol{a}_2 + \langle\ldots\rangle\boldsymbol{a}_3$$

‡Equation numbers followed by the letter G refer to equations in Dr. Grubin's Comment.

A glance at Eqs. (9KL) and (10KL) reveals that Dr. Grubin's procedure leads directly and precisely to Eq. (9KL) rather than Eq. (10KL). Consequently, Dr. Grubin's claim at the end of his elaboration on point 1 to the effect that "Eq. (9KL) and its counterparts never appear" is ill-founded. Moreover, his contention that "it is not necessary to locate explicitly the combined mass center (CMC) for the entire system" must give one pause in light of his later injunction to equate the inertial derivative of $\bar{H}_{\mathrm{CMC}}$ to "the vector torque around the CMC." How is one to do this without first locating the CMC explicitly?

2) Dr. Grubin's statement "The origin for the angular momentum equation is not restricted to the CMC; any origin..." gives rise, at best, to a question of semantics: What is *the* angular momentum equation? It is well known that, at times, one can work effectively with angular momenta relative to points other than the mass center of a system, e.g., when one is dealing with a system possessing an inertially fixed point other than the mass center. An equation involving such an angular momentum is no more and no less *the* angular momentum equation than is any other that can be deduced from it or from which it can be deduced.

In the second sentence of point 2, Dr. Grubin speaks of "proof of D'Alembert's principle," and in elaborating he states that his "generalized angular momentum equation" leads to the equations formulated in our paper with the aid of D'Alembert's principle. This is misleading for two reasons. First, it tends to create the impression that D'Alembert's principle requires proof, which is incorrect, for it is just as reasonable to deduce momentum principles from D'Alembert's principle as it is to proceed vice-versa. Second, Dr. Grubin's statement obscures a substantative matter—namely, that the use of D'Alembert's Principle in the way set forth in our paper results in a savings of labor because it makes it unnecessary to perform the often arduous differentiations of angular momenta, such as the one indicated in Eq. (10G). Instead, it permits one to use familiar expressions for inertia torques and inertia forces, which can be formed as soon as one has completed the requisite kinematical analyses. Here again, it is most helpful to work out a particular problem in *all detail* if one seeks to acquire clear ideas regarding the relative merits of two methods for formulating equations of motion.